Discussion 5 Worksheet

Vector-valued functions and partial derivatives

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MATH 53 Multivariable Calculus

1 Integrals of Vector Functions

1. Evaluate $\int_1^2 \langle \frac{\ln(t)}{t}, e^{-t} \rangle dt$.

2. Suppose that $\vec{r''}(t) = \langle 6t, \sin(t) \rangle$ and it is known that $\vec{r'}(0) = \langle 1, -1 \rangle$ and $\vec{r}(0) = \langle 0, 1 \rangle$. Find a formula for $\vec{r}(t)$.

2 Vector Function Basics

(a) Find the limit

$$\lim_{t \to 0} \left\langle e^{-3t}, \frac{t^2}{\sin^2 t}, \cos 2t \right\rangle.$$

(b) Find the limit

$$\lim_{t \to \infty} \left\langle \frac{1+t^2}{1-t^2}, \arctan t, \frac{1-e^{-2t}}{t} \right\rangle.$$

(c) Find a vector equation and parametric equations for the line segment that joins $(2, 0, 0)$ to $(6, 2, -2)$.

(d) Find a vector equation and parametric equations for the line segment that joins $(1, 5, 6)$ to $(3, 1, 8)$.

(e) Find a vector function that represents the curve of the intersection of the cone $z = \sqrt{x^2 + y^2}$ and $z = 1 + y$.

(f) Suppose the trajectories of two particles are given by

$$r_1(t) = \langle t^2, 7t - 12, t^2 \rangle \qquad r_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geq 0$. Do the particles collide?

3 Challenge: Vector Orthogonality

(a) Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

4 Graphs of Multivariable Functions

Sketch the graph of the function.

(a) $f(x, y) = y$;

(b) $f(x, y) = 10 - 4x - 5y$;

(c) $f(x,y) = \sin x$;

(d) $f(x,y) = \sqrt{4-4x^2-y^2}$.

5 Evaluating Partial Derivatives

1. Compute the following partial derivatives:

   $\frac{\partial}{\partial x}(x^2 e^{xy})$.

   $\frac{\partial^{10}}{\partial x^{10}}\left(\frac{\partial^{13}}{\partial y^{13}}\left(x^{10}y^{13}\right)\right)$.

   $\frac{\partial}{\partial w}(\sin(w\sin(wv)))$.

   $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{e^{xy\sin(y)-x}}{y\sin(y)-1}\right)\right)$.Hint: Clairaut's Theorem simplifies the calculation.

2. In the equation $PV = T$, any one of the three variables can be solved for as a function of the other two. Show that $\left(\frac{\partial P}{\partial T}\right)\left(\frac{\partial T}{\partial V}\right)\left(\frac{\partial V}{\partial P}\right) = -1$.

6 More on Partial Derivatives

1. Suppose that the values of a function $f(x,y)$ at four points are given by the following table:

	x=1.3	x=1.4
y=0.4	2.3	2.5
y=0.6	1.5	1.4

   Estimate $f_x(1.3, 0.4)$ and $f_x(1.3, 0.6)$. Then estimate $f_{xy}(1.3, 0.4)$.

2. Consider a smooth function $f(x,y)$ of two variables. List all possible third order partial derivatives of f. Your list should not contain two equivalent expressions. Here “smooth” means that all relevant derivatives of f exist and are continuous.

3. Suppose that the partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$ exist. If f_x is the zero function, show that f is a function of y only. More precisely, there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(y)$ for all real numbers x and y.

   Hint: For fixed y, consider the function $h(t) = f(t,y)$ and differentiate.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.